MTH 403: Real Analysis II

Practice Assignment III

Conceptual problems

1. Try Problems 4-13, 4-14, 4-18, and 4-19 (page 96) from *Calculus on manifolds* by M. Spivak.

2. Recalling your solution to Problem 3-41 (from Practice Assignment - II), try Problem 4-21 (page 97) from *Calculus on manifolds* by M. Spivak.

3. Try Problems 4-25, 4-28 and 4-33 (pages 104-105) from *Calculus on manifolds* by M. Spivak.

Integration on chains

1. If $\omega \in \Lambda^n(V)$ is the volume element determined by T and μ, and $w_1, \ldots, w_n \in V$, then show that
$$\omega(w_1, \ldots, w_n) = \sqrt{\det(g_{ij})},$$
where $g_{ij} = T(w_i, w_j)$.

2. If $\omega \in \Lambda^n(V)$ is the volume element determined by T and μ, and $f : \mathbb{R}^n \to V$ is an isomorphism such that $f^*T = \langle , \rangle$ and $[f(e_1), \ldots, f(e_n)] = \mu$, then show that $f^*\omega = \det$.

3. Show that every non-zero $\omega \in \Lambda^n(V)$ is the volume element determined by some inner product T and orientation μ for V.

4. If $w_1, \ldots, w_{n-1} \in \mathbb{R}^{n-1}$, then show that
$$|w_1 \times \ldots \times w_{n-1}| = \sqrt{\det(g_{ij})},$$
where $g_{ij} = \langle w_i, w_j \rangle$.

5. If $f : U \to \mathbb{R}^n$ be a differentiable function with a differentiable inverse $f^{-1} : f(U) \to \mathbb{R}^n$. If every closed form on U is exact, show that the same is true for $f(U)$.

6. For $R > 0$ and $n \in \mathbb{Z}$, define a singular 1-cube $C_{R,n} : [0,1] \to \mathbb{R}^2 - \{0\}$ by $C_{R,n}(t) = (R\cos(2\pi nt), R\sin(2\pi nt))$. Then prove the following statements.
   (a) There is a singular 2-cube $c : [0,1]^2 \to \mathbb{R}^2 - \{0\}$ such that $C_{R_1,n} - C_{R_2,n} = \partial c$.
   (b) $\int_{C_{R,n}} d\theta = 2\pi n$ and that there exists no 2-chain c in $\mathbb{R}^2 - \{0\}$ such that $C_{R,n} = \partial c$. [Hint: Use Stokes' Theorem.]

7. If c is a singular 1-cube in $\mathbb{R}^2 - \{0\}$ with $c(0) = c(1)$, show that there is a unique integer n such that $c - c_{1,n} = \partial c^2$ for some 2-chain c^2. (The integer n is called the *winding number* of c around 0.)

8. If ω is a 1-form fdx on $[0,1]$ with $f(0) = f(1)$, show that there exists a unique number λ such that $\omega - \lambda dx = dg$ for some g with $g(0) = g(1)$.

9. If ω is a 1-form on $\mathbb{R}^2 - \{0\}$ such that $d\omega = 0$, prove that $\omega = \lambda d\theta + dg$, for some $\lambda \in \mathbb{R}$ and $g : \mathbb{R}^2 - \{0\} \to \mathbb{R}$.